Estimating extinction time from the fossil record

using regression inversion

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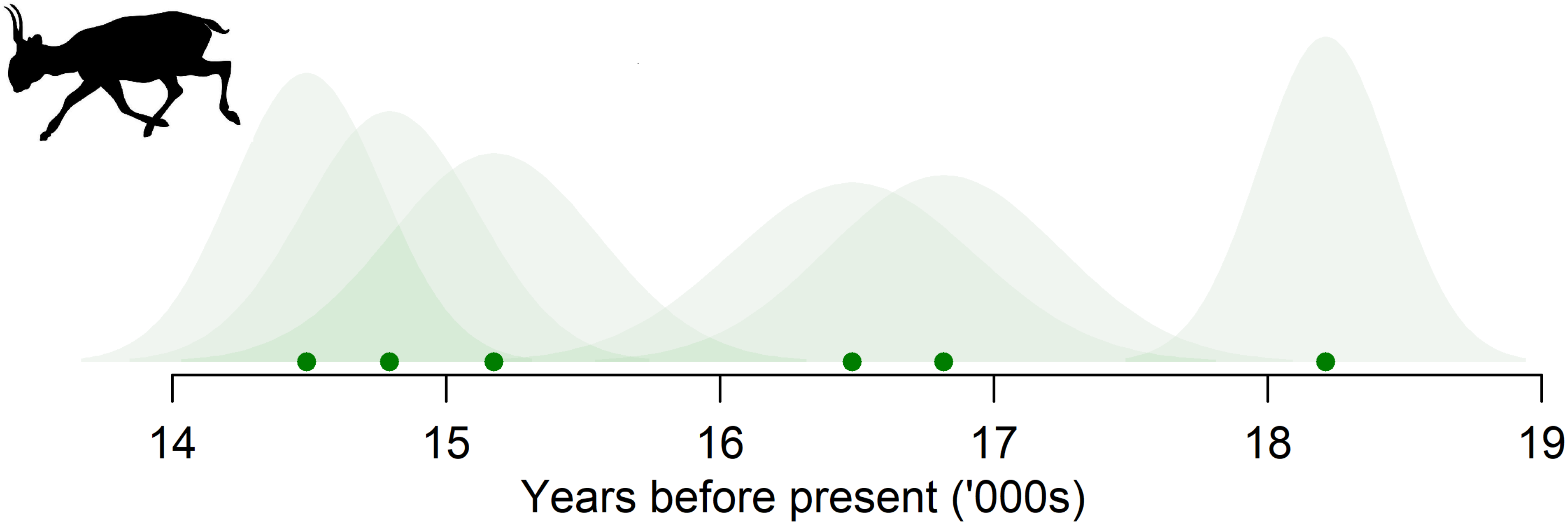
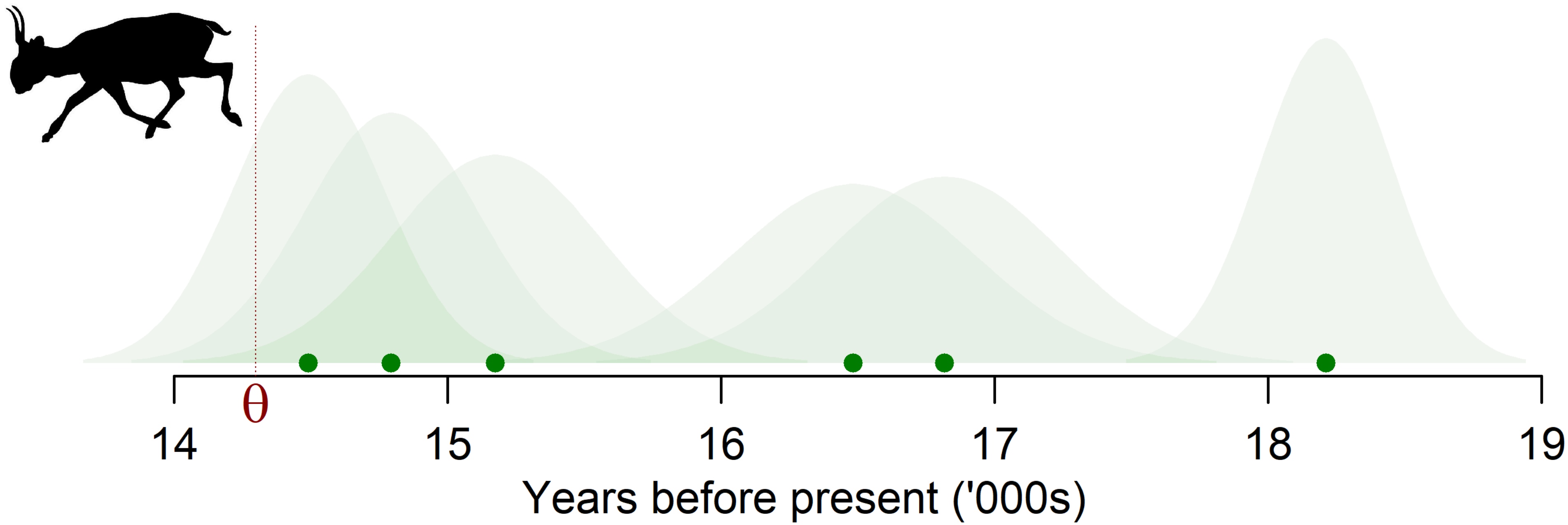
Estimating extinction time from the fossil record



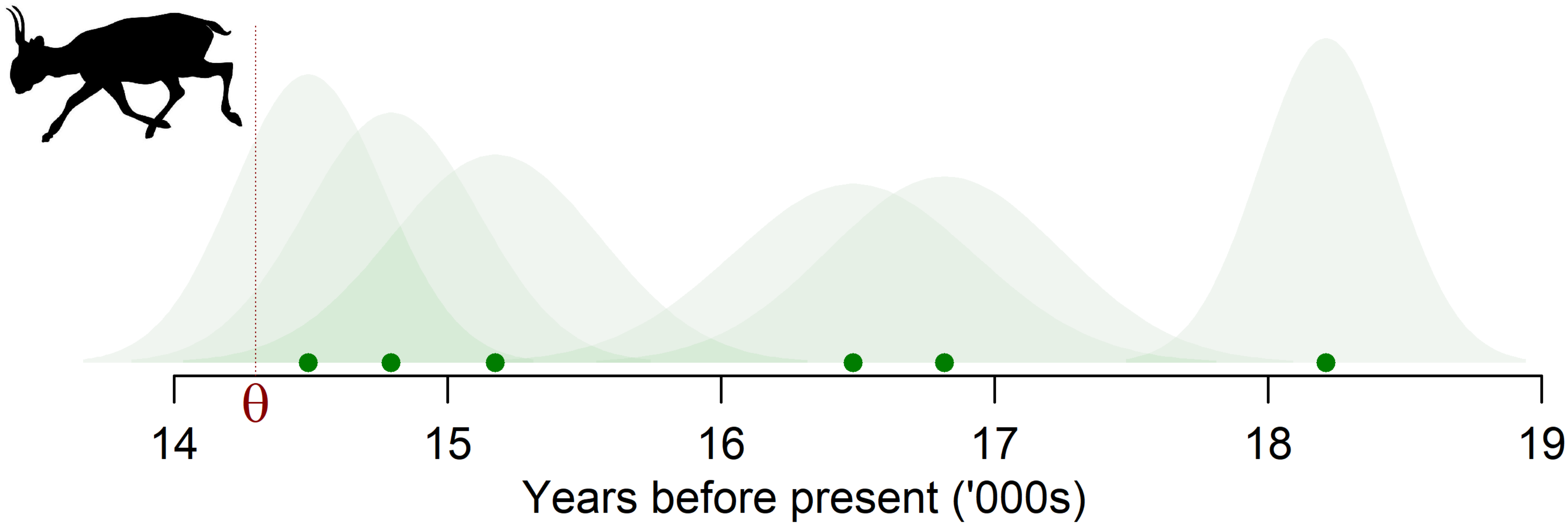
- We have a set of specimens from the fossil record with estimated ages
- Ages are measured with a known amount of error

Estimating extinction time from the fossil record



- We have a set of specimens from the fossil record with estimated ages
- Ages are measured with a known amount of error
- We want to estimate extinction time θ

Key data properties



We need to deal with

- *Sampling error* – last time it was seen isn't the last time it was there
- *Measurement error* – ages are estimated with error

Previous work

Who	Sampling error	Measurement error	Small samples
Strauss and Sadler (1989)	✓	×	✓
`GRIWM` by Bradshaw et al. (2012)	×	✓	✓
Solow, Roberts, and Robbirt (2006)	✓	✓	×

Aim

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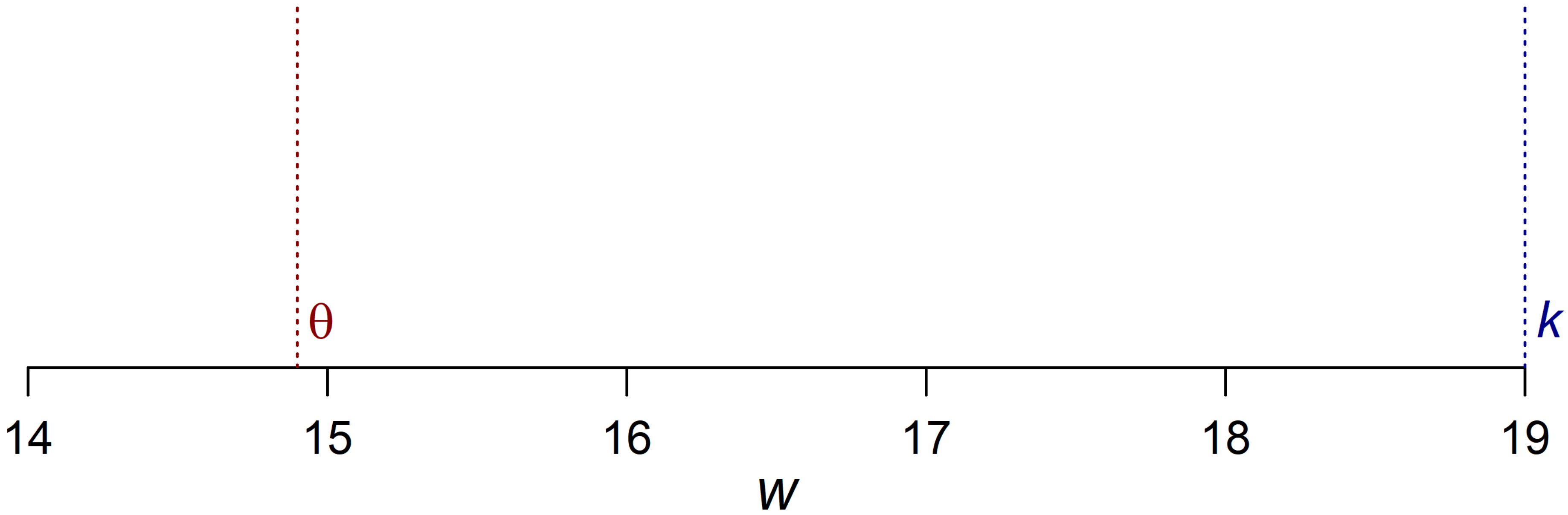
Solow, Roberts, and Robbirt (2006)	✓	✓	×
This talk	**✓**	**✓**	**✓**

The Compound Uniform-Truncated T Distribution

We observe $W = X + \epsilon$ which can be no larger than k.

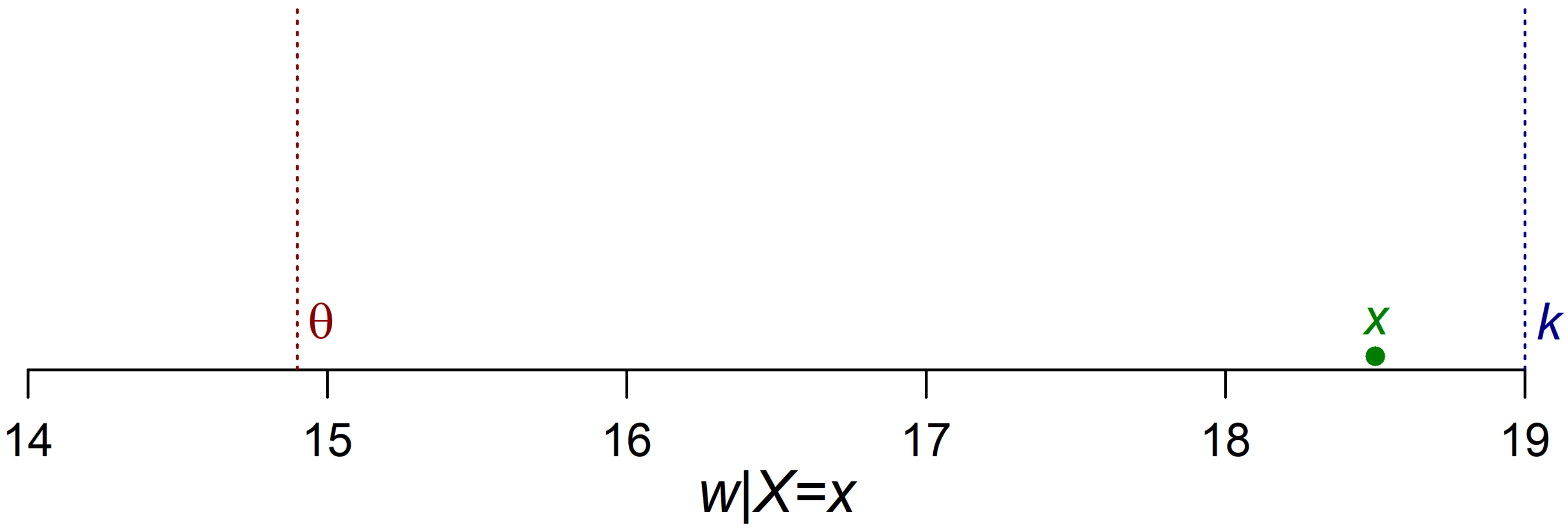


X is the true fossil age, which can be no smaller than θ.

ϵ is measurement error with known standard deviation σ.

The Compound Uniform-Truncated T Distribution

We propose defining a distribution for W via the conditionals of (X, ϵ):



- Conditional on $X = x \ldots$

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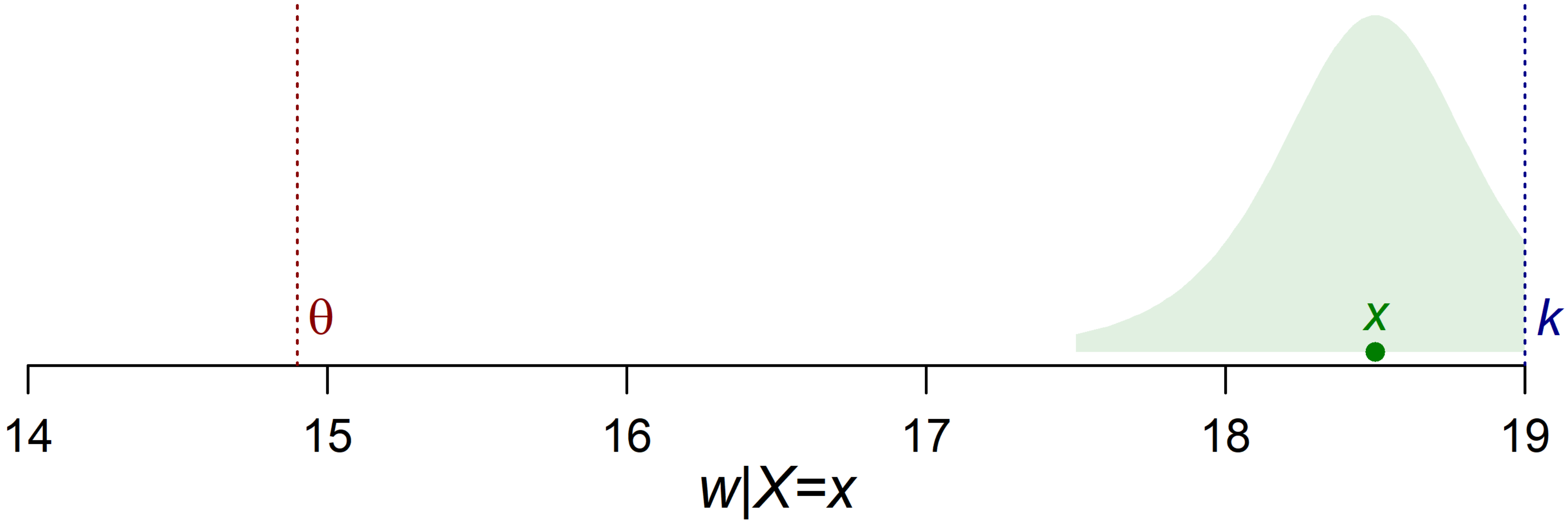

- Conditional on $X = x$, we assume $\epsilon | X \stackrel{d}{=} (T | T < k - x)$, where $T \sim t_\nu$

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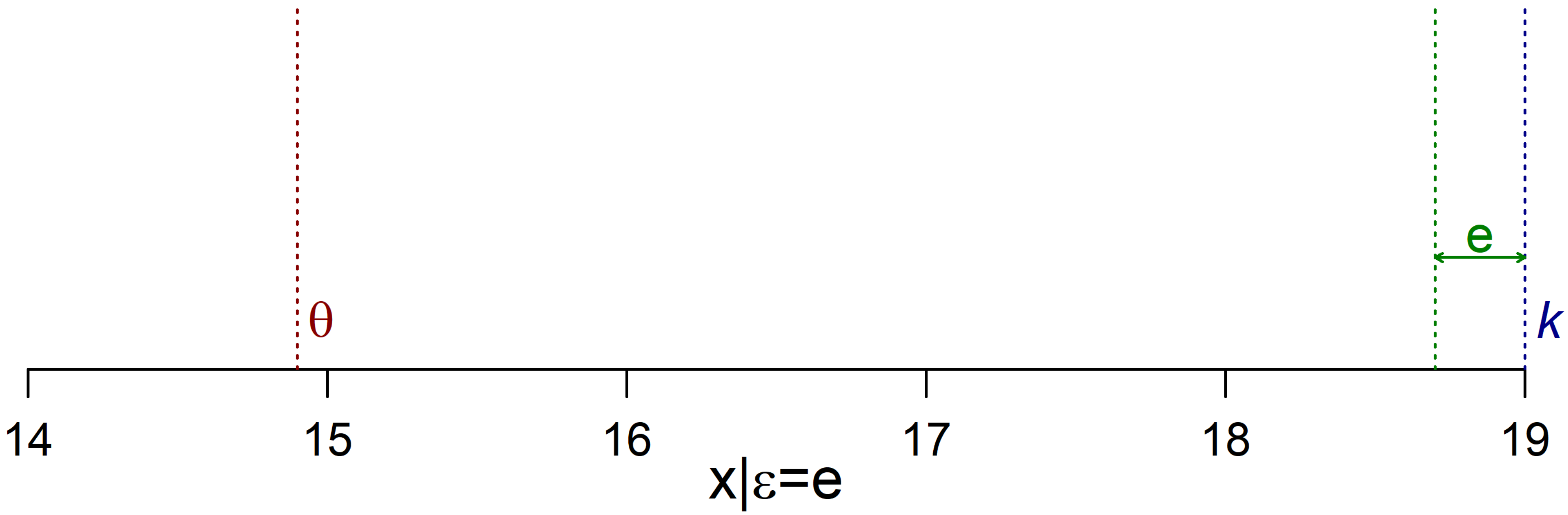

- Conditional on $X = x$, we assume $\epsilon | X \stackrel{d}{=} (T | T < k - x)$, where $T \sim t_\nu$
- Conditional on $\epsilon = e \ldots$

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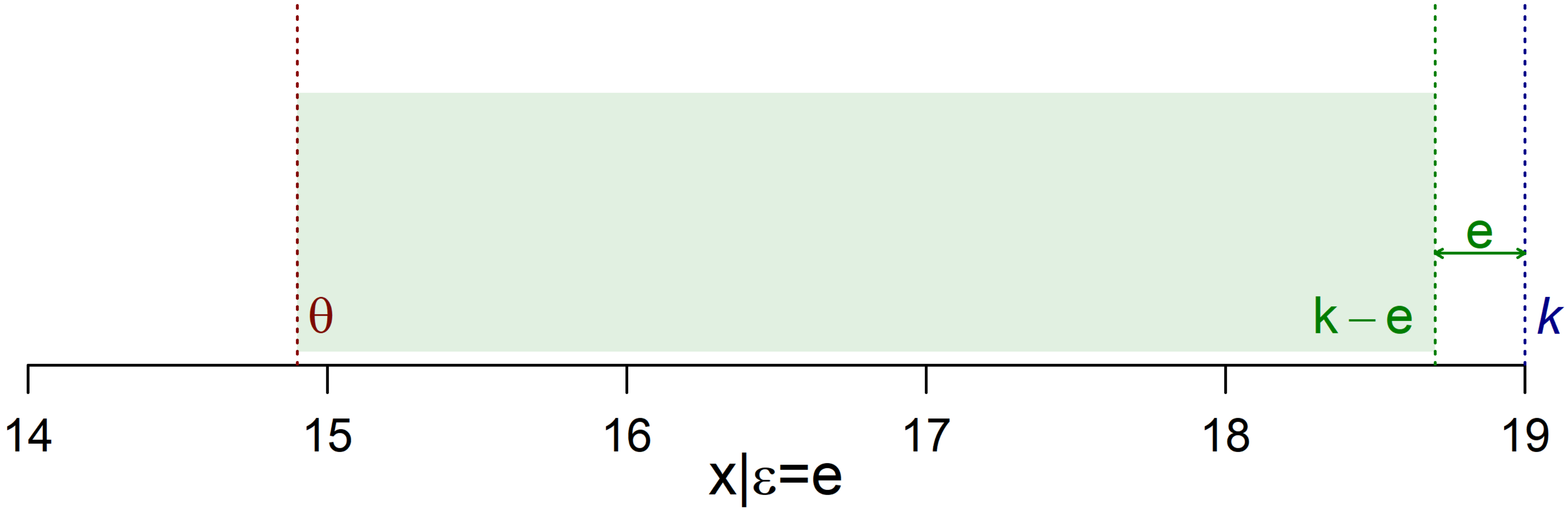


- Conditional on $X = x$, we assume $\epsilon | X \stackrel{d}{=} (T | T < k - x)$, where $T \sim t_\nu$
- Conditional on $\epsilon = e$, we assume $X | \epsilon \sim \mathcal{U}[\theta, k - e]$

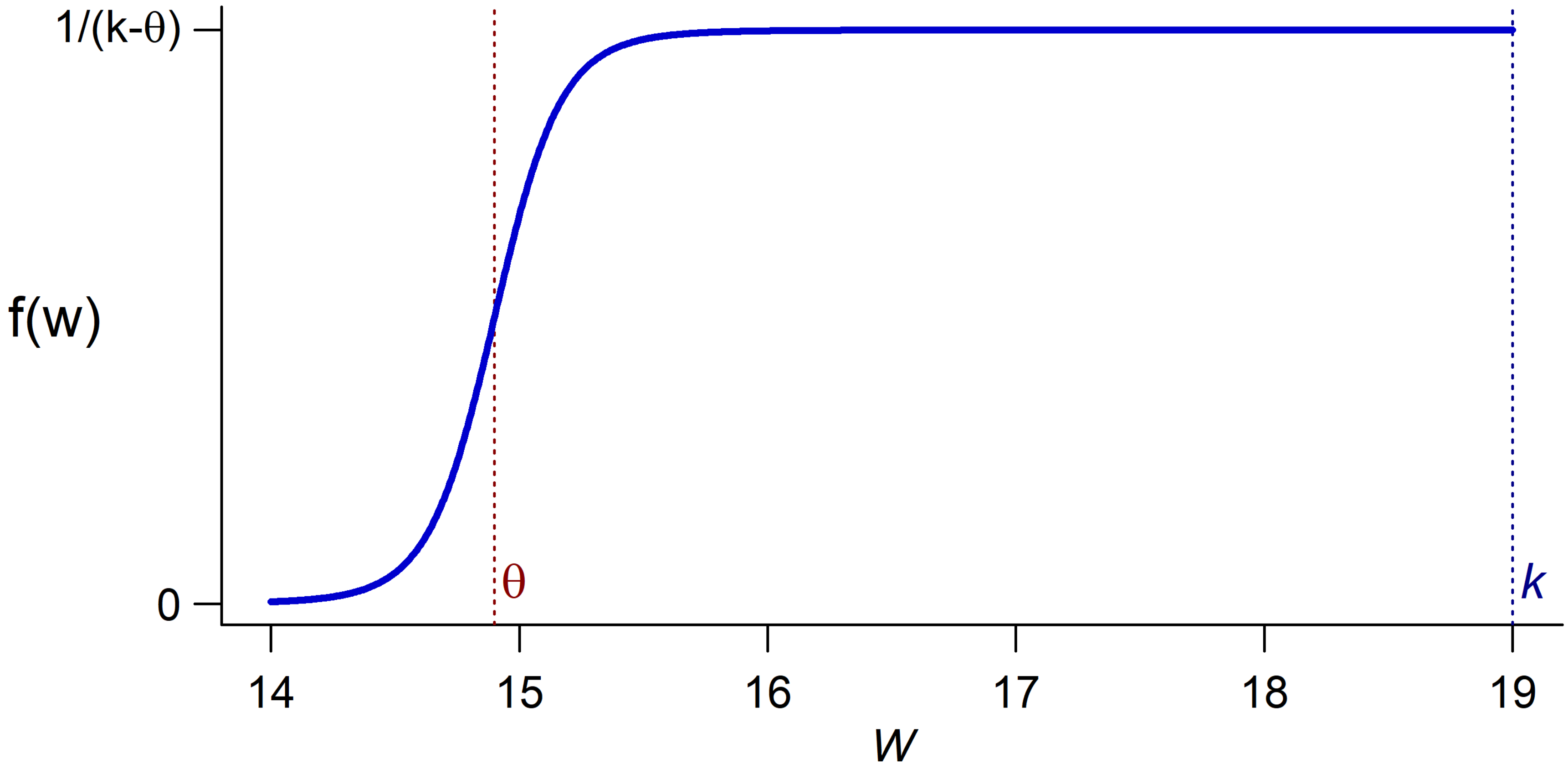
The Compound Uniform-Truncated T Distribution

We can then show that the probability density function of W is:

$$f(w) = \frac{G_\nu\left(\frac{w-\theta}{\sigma}\right)}{(k-\theta)G_\nu\left(\frac{k-\theta}{\sigma}\right) + \sigma g_\nu\left(\frac{k-\theta}{\sigma}\right)\left(\frac{\nu}{\nu-1}\right)\left(1+\frac{(k-\theta)^2}{\sigma^2\nu}\right)}$$

where $G_\nu(\cdot)$ and $g_\nu(\cdot)$ are the cumulative distribution function and the probability density function of Student's t_ν distribution.

The Compound Uniform-Truncated T Distribution



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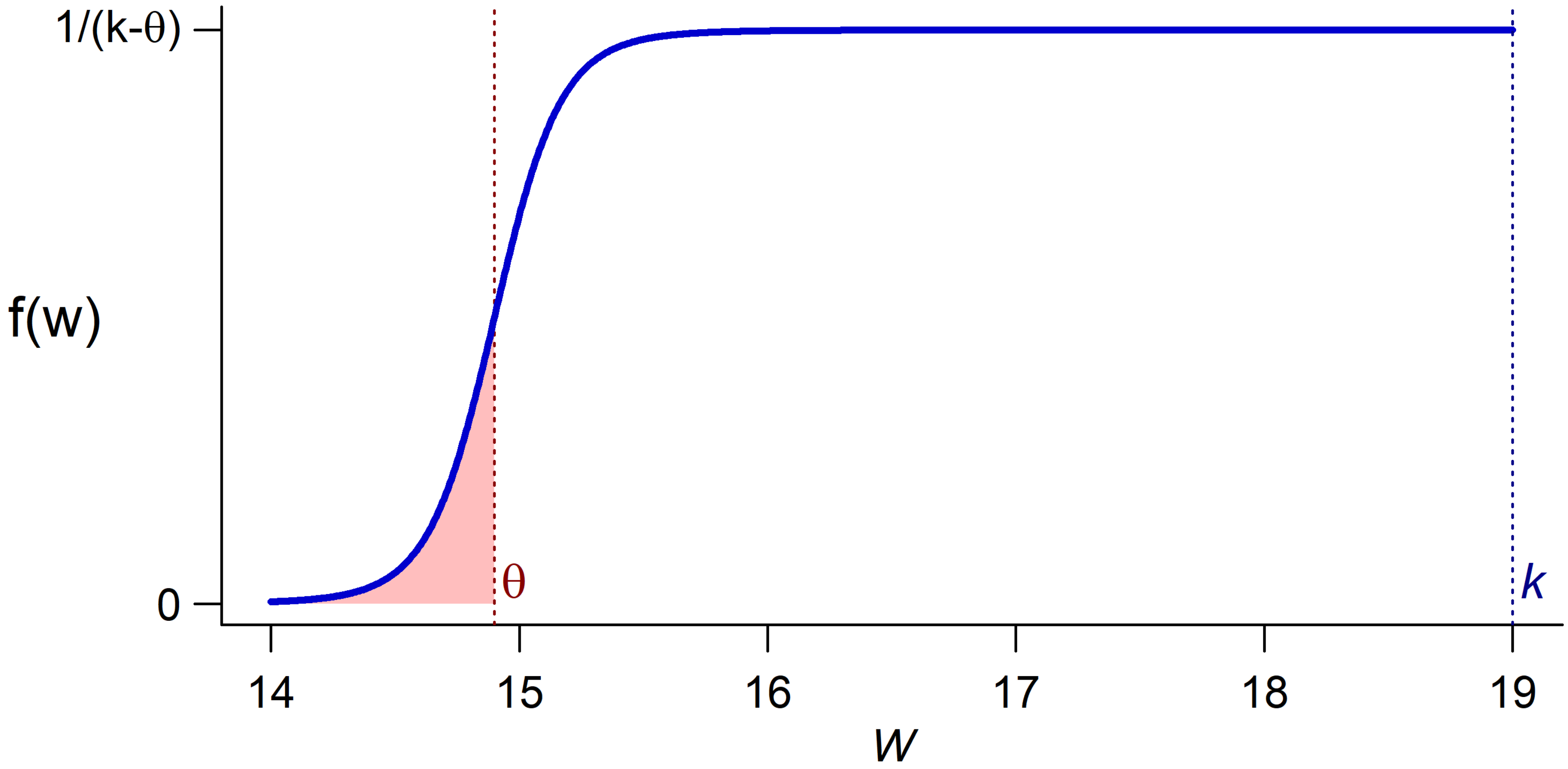
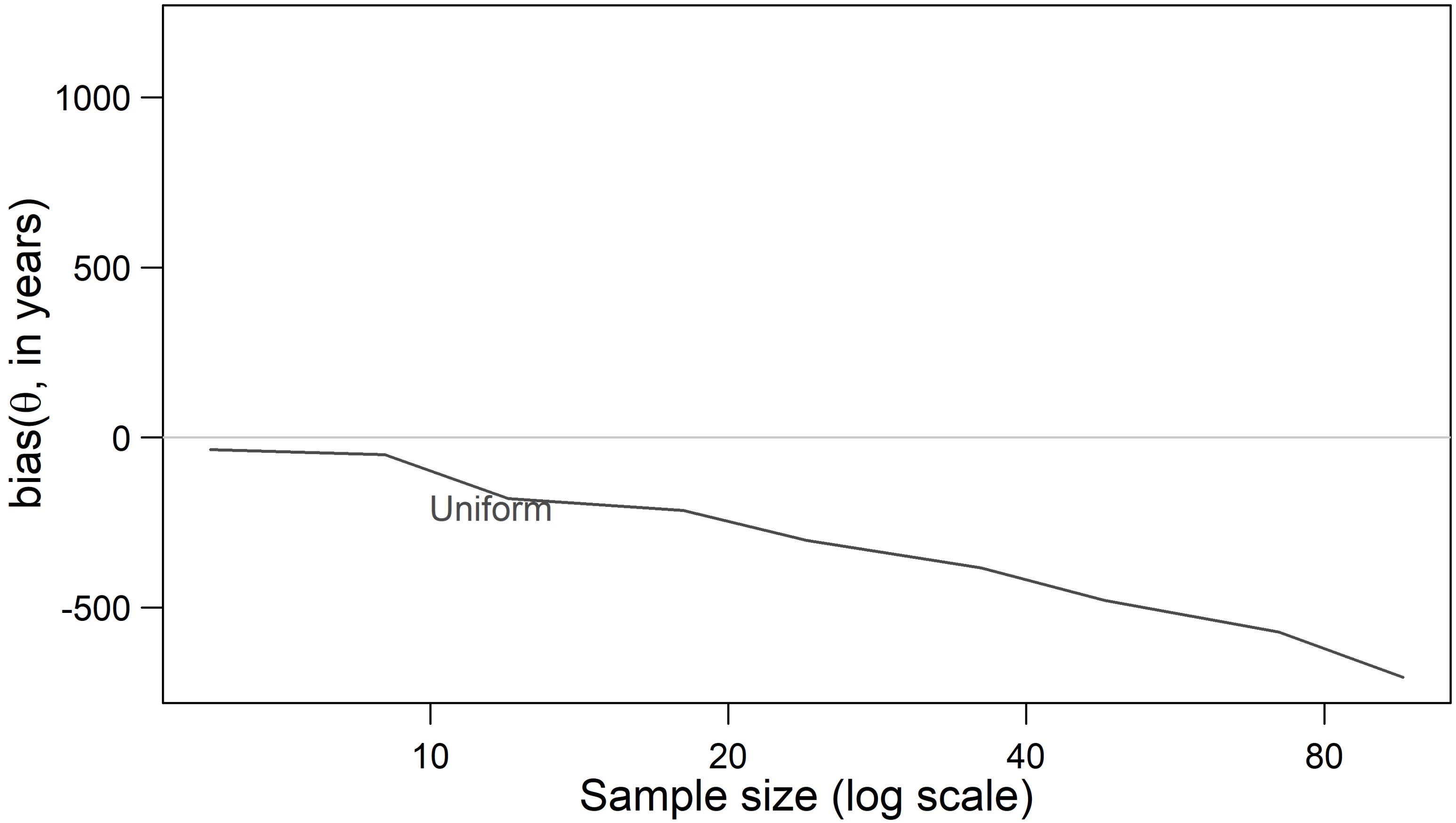


Some observed ages will be *younger* than true time of extinction!

Ignore measurement error $\rightarrow$ negative bias



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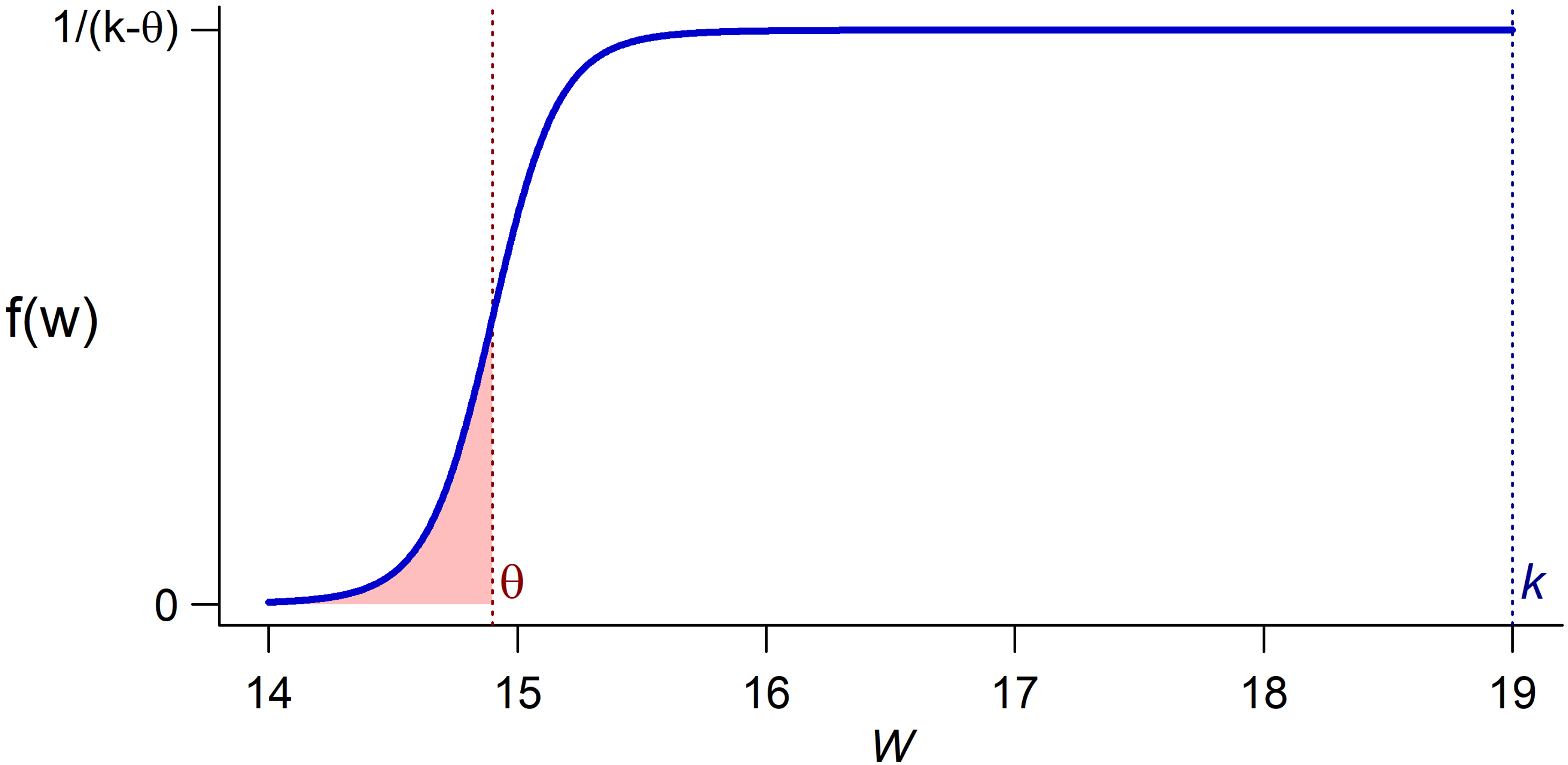

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Asymptotic Inference about θ?

1. Wald CI:

$$\left[\hat{\theta} - \frac{I(\hat{\theta})^{-1}}{\sqrt{n}}, \hat{\theta} + \frac{I(\hat{\theta})^{-1}}{\sqrt{n}}\right]$$

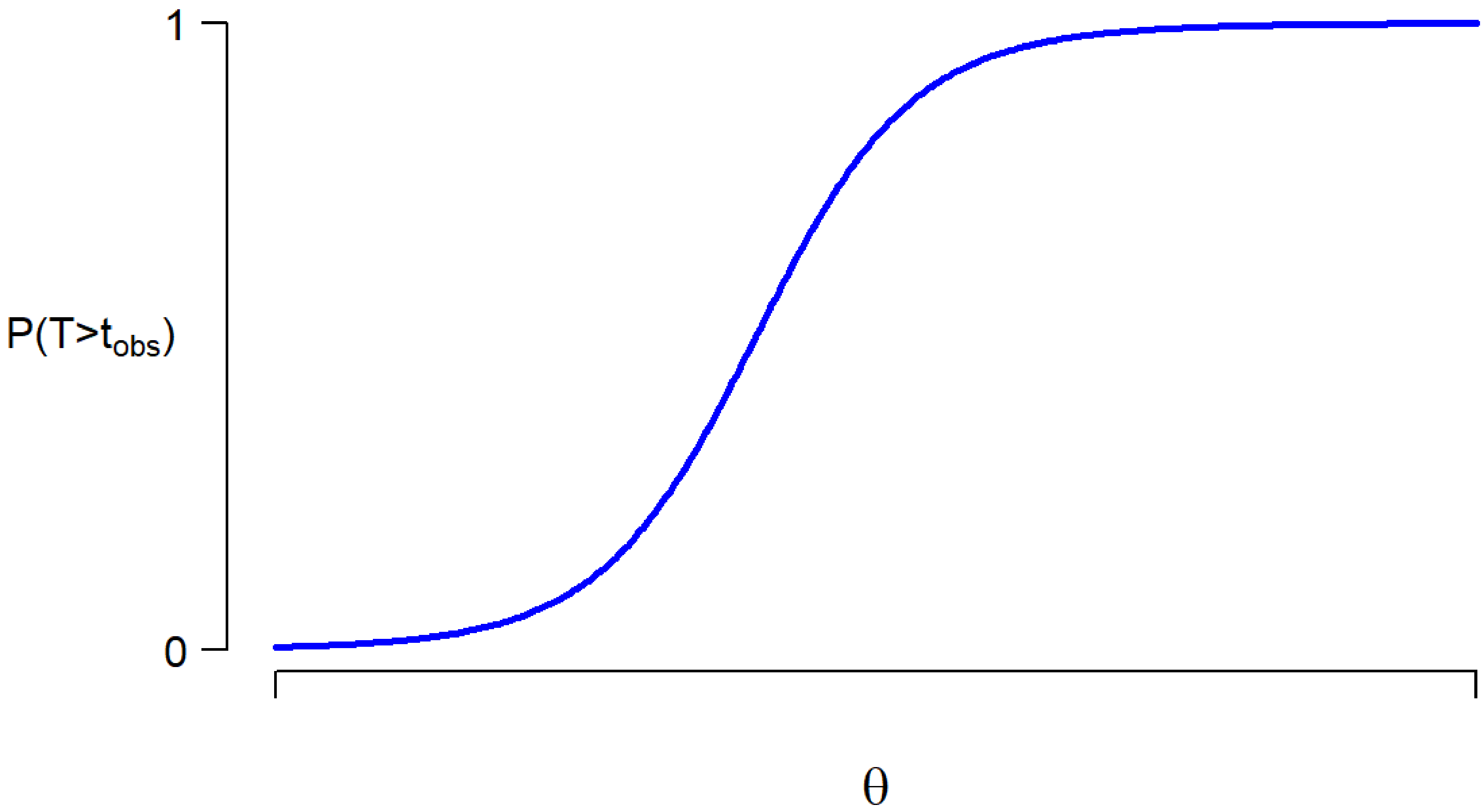
   since $\sqrt{n}I(\theta_0)(\hat{\theta} - \theta_0) \xrightarrow{d} \mathcal{N}(0,1)$

2. Likelihood ratio test inversion:

$$\{\theta : -2\log\Lambda(\theta) < q_\alpha\}$$

   where $\mathbb{P}(X^2 > q_\alpha) = \alpha$ for $X^2 \sim \chi^2_1$, since
   $-2\log\Lambda(\theta_0) = 2\ell(\hat{\theta}) - 2\ell(\theta_0) \xrightarrow{d} \chi^2_1$

Confidence intervals by inversion



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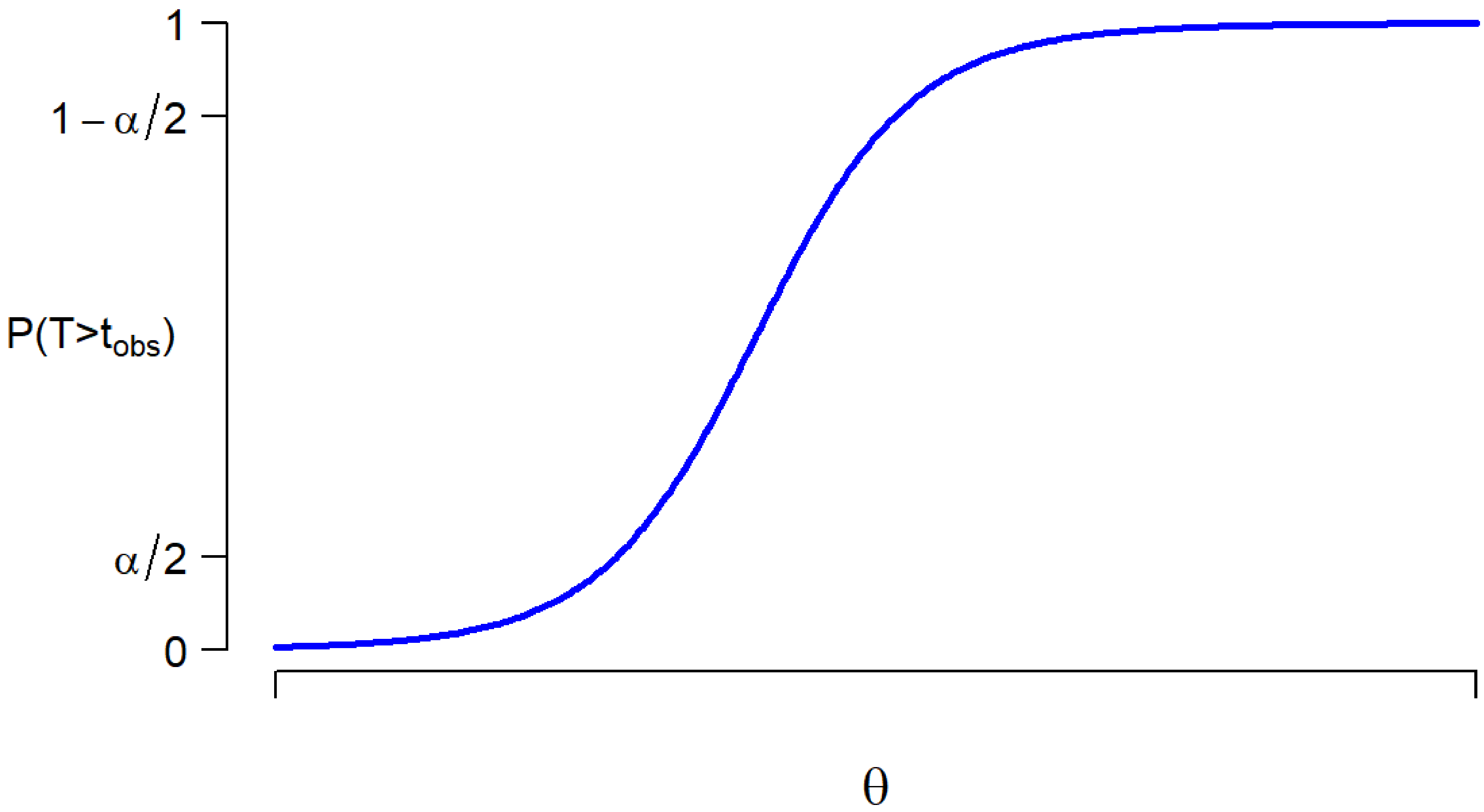

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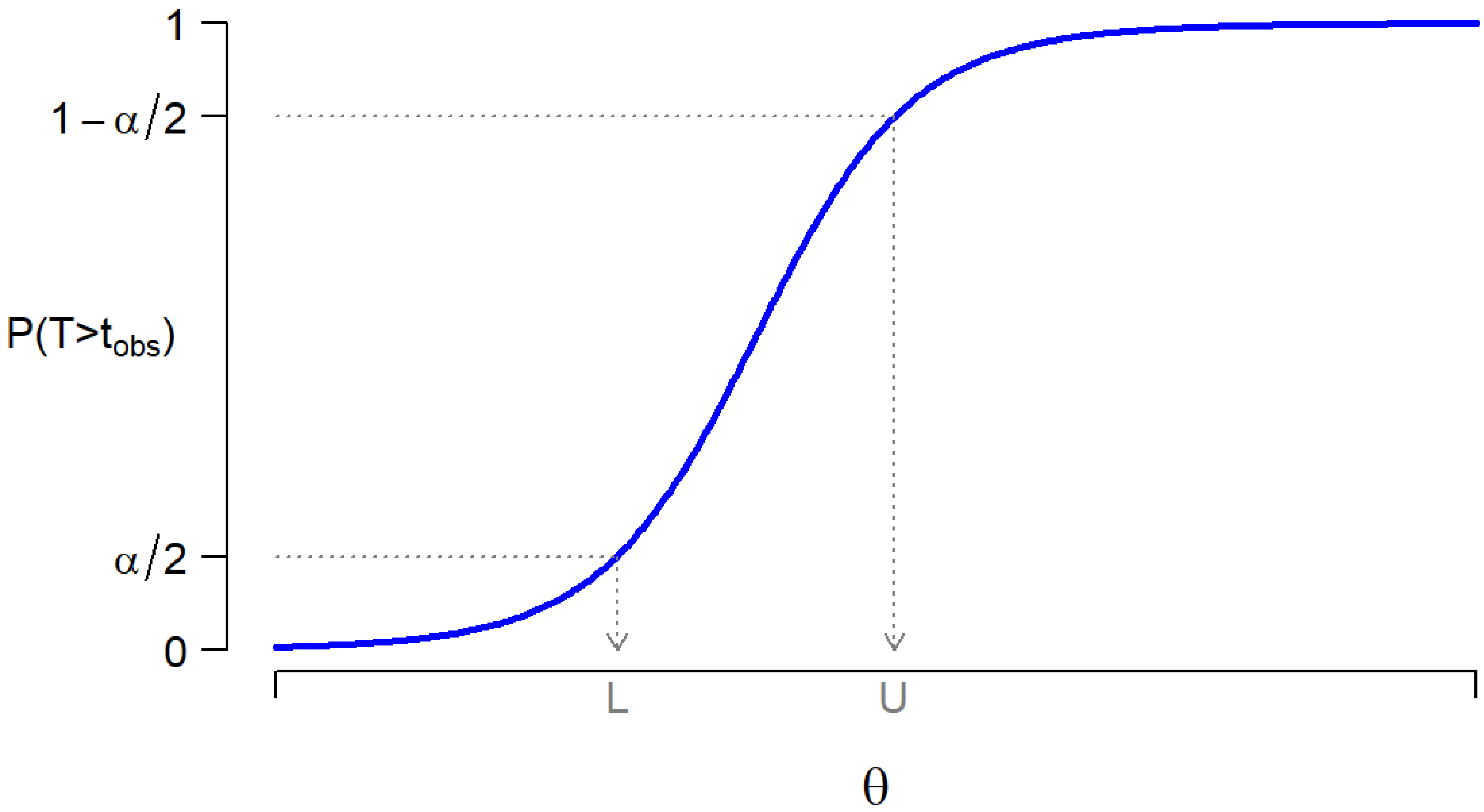

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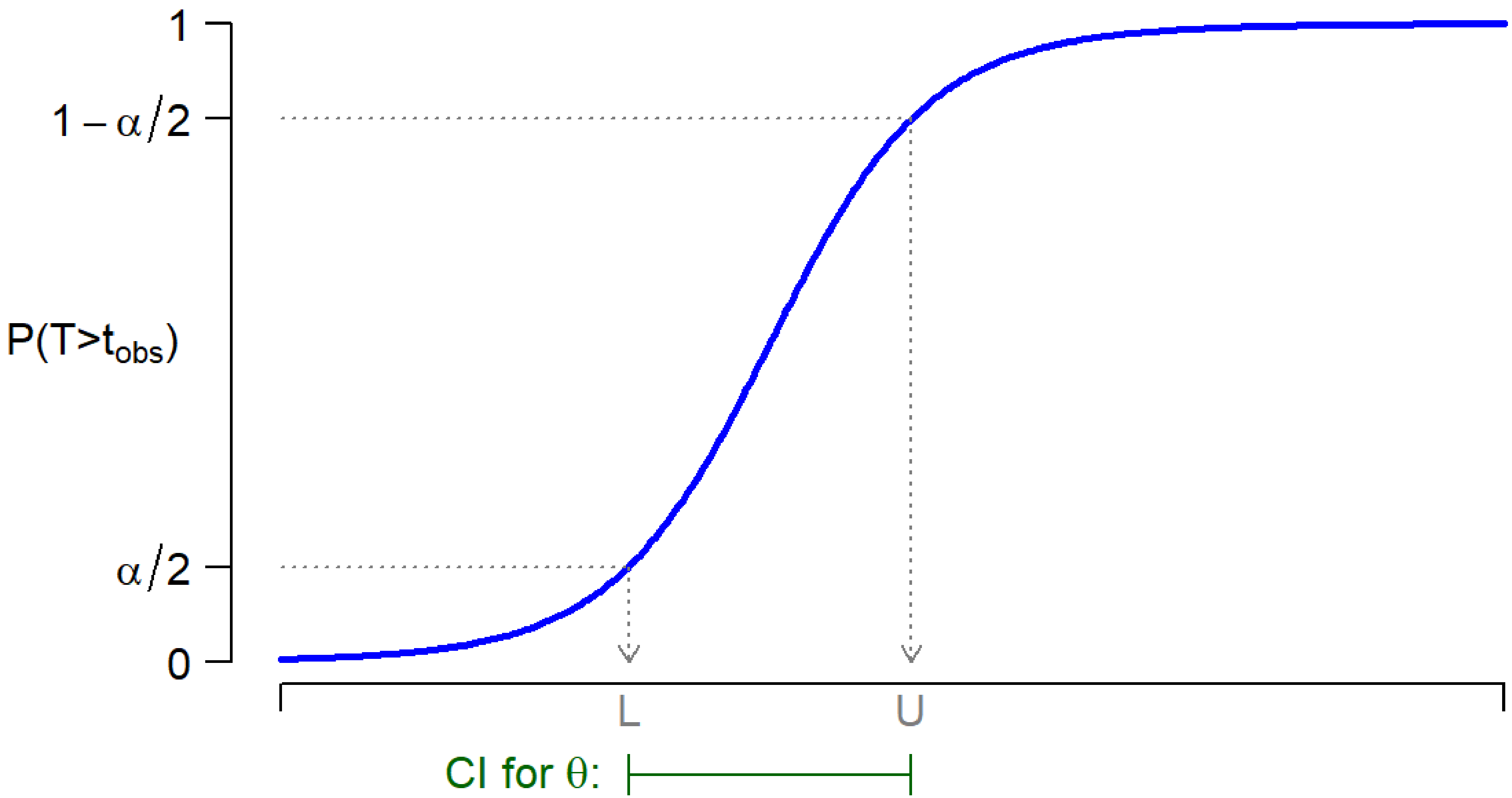

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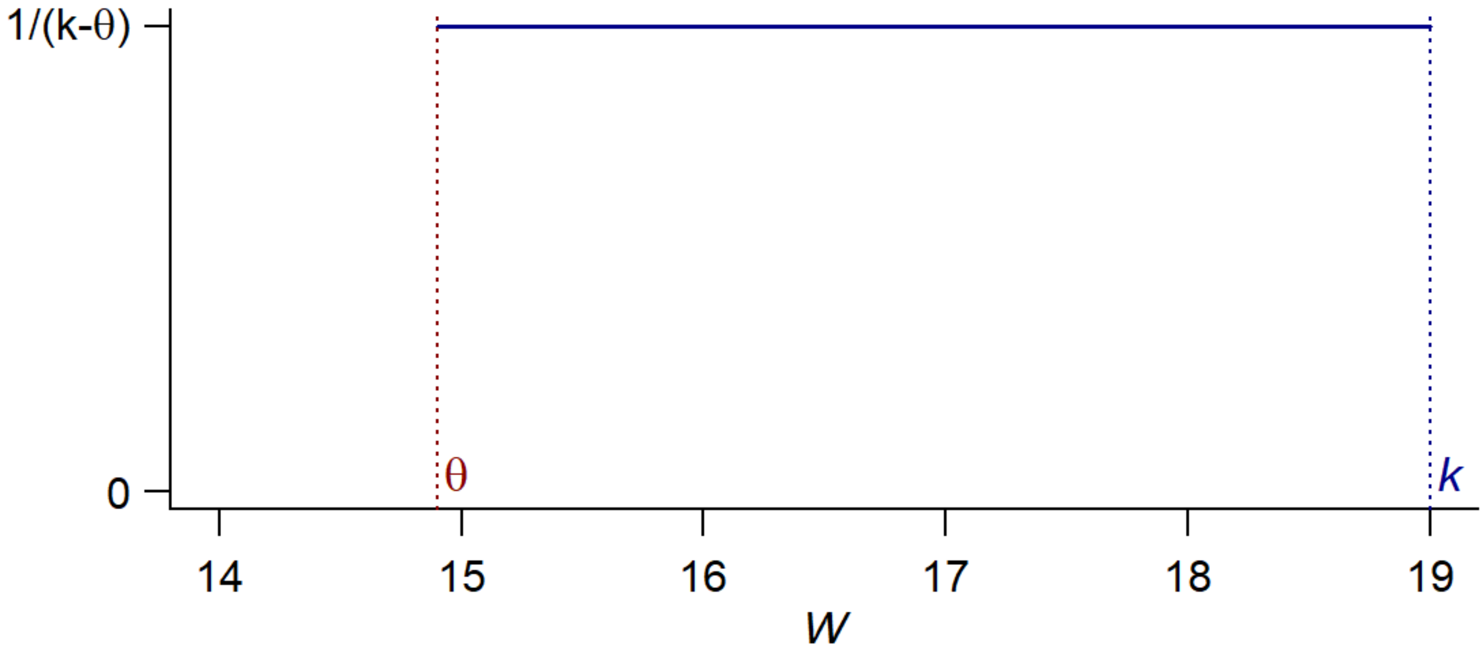
Problem: doesn't work for small σ (or small n)

We are in the neighbourhood of $\sigma = 0$



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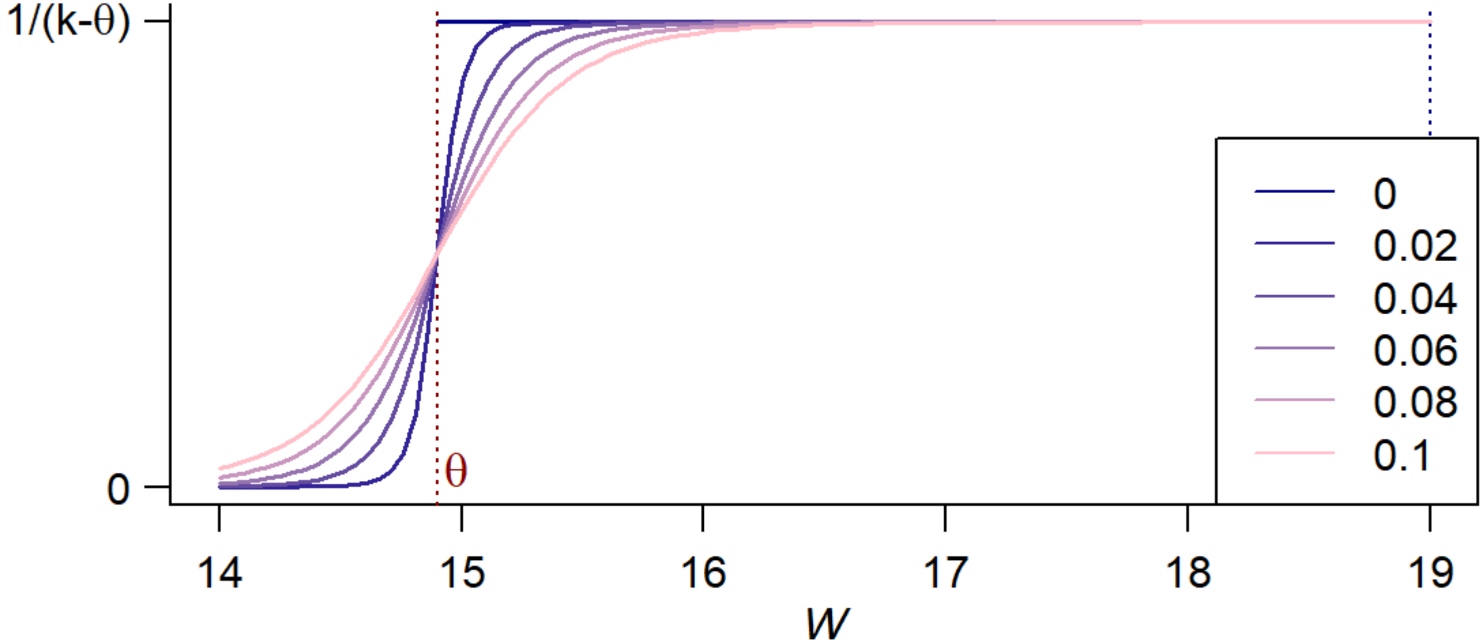

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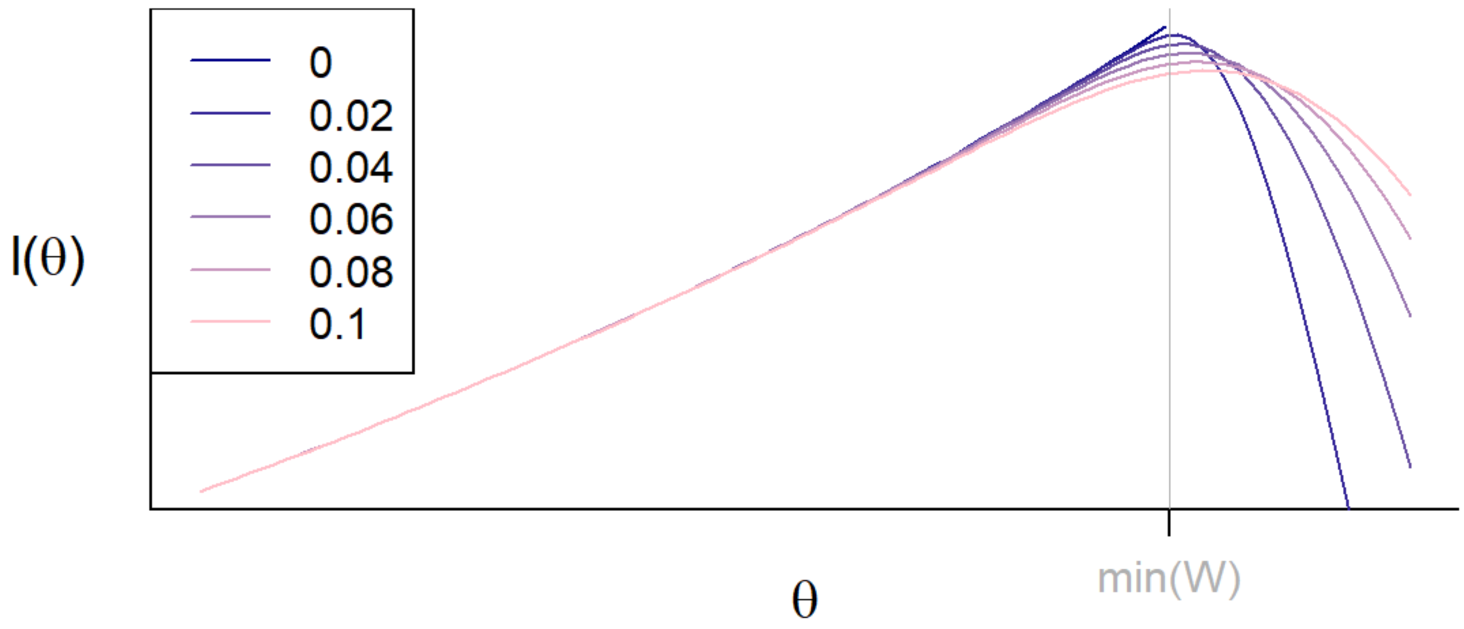

- the likelihood is not smooth

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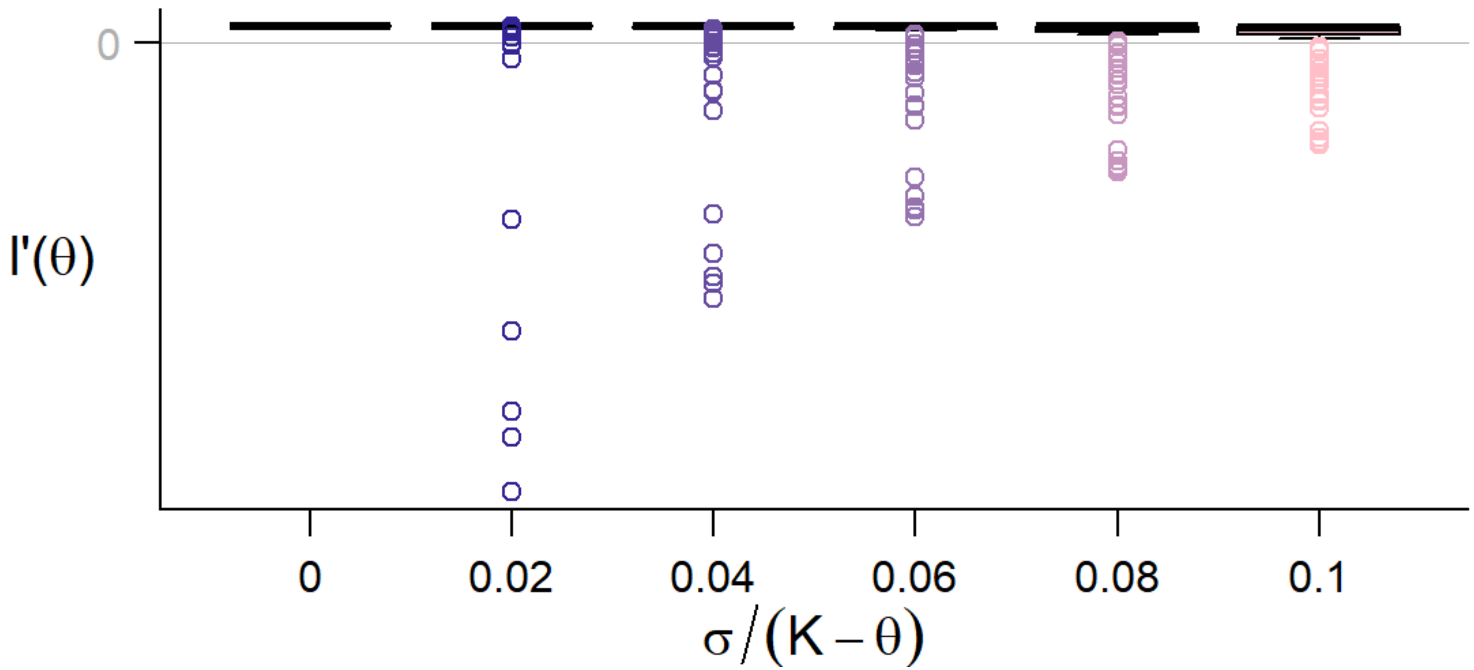
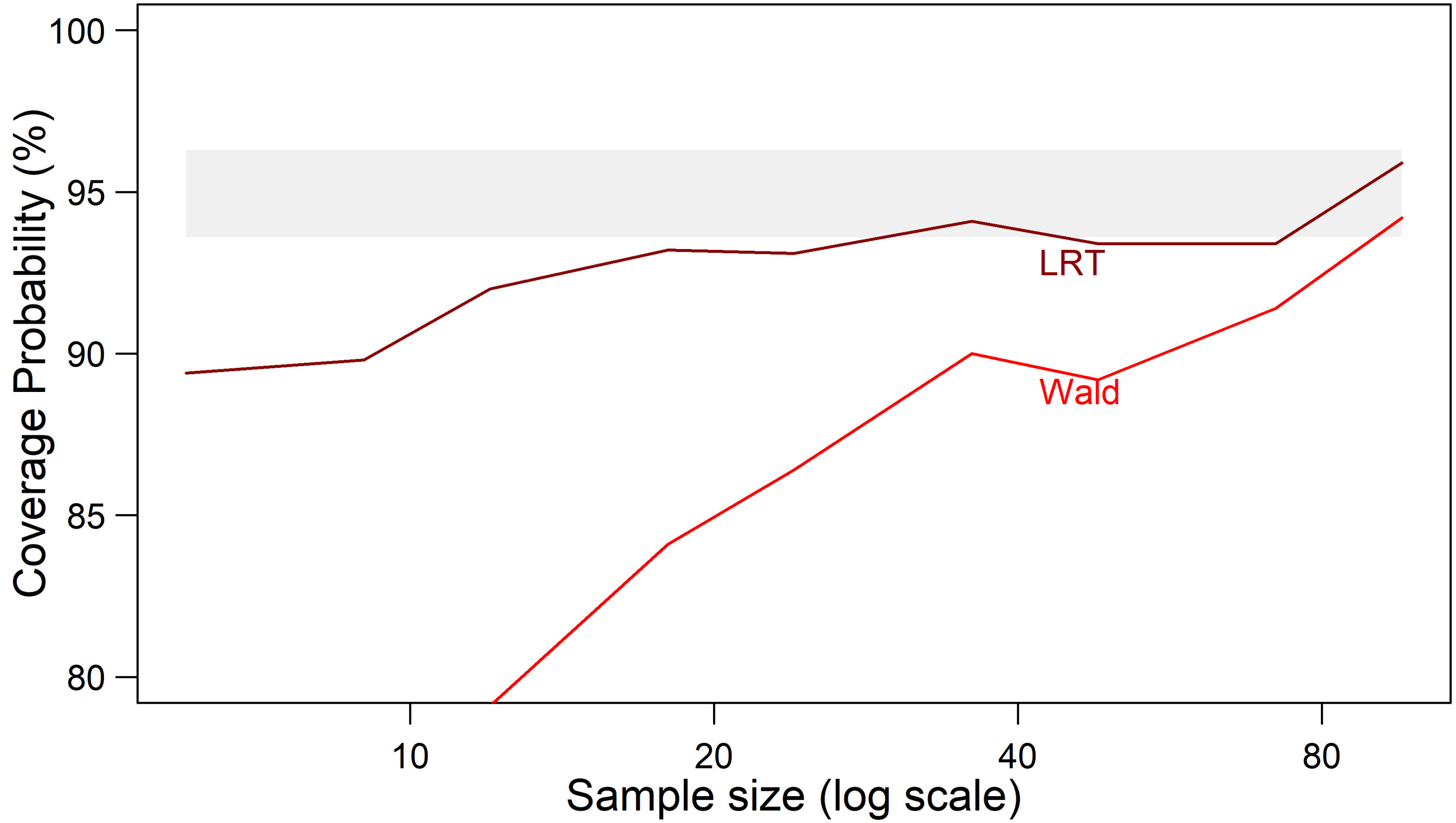


- the likelihood is not smooth
- Central Limit Theorem does not apply to the score equation

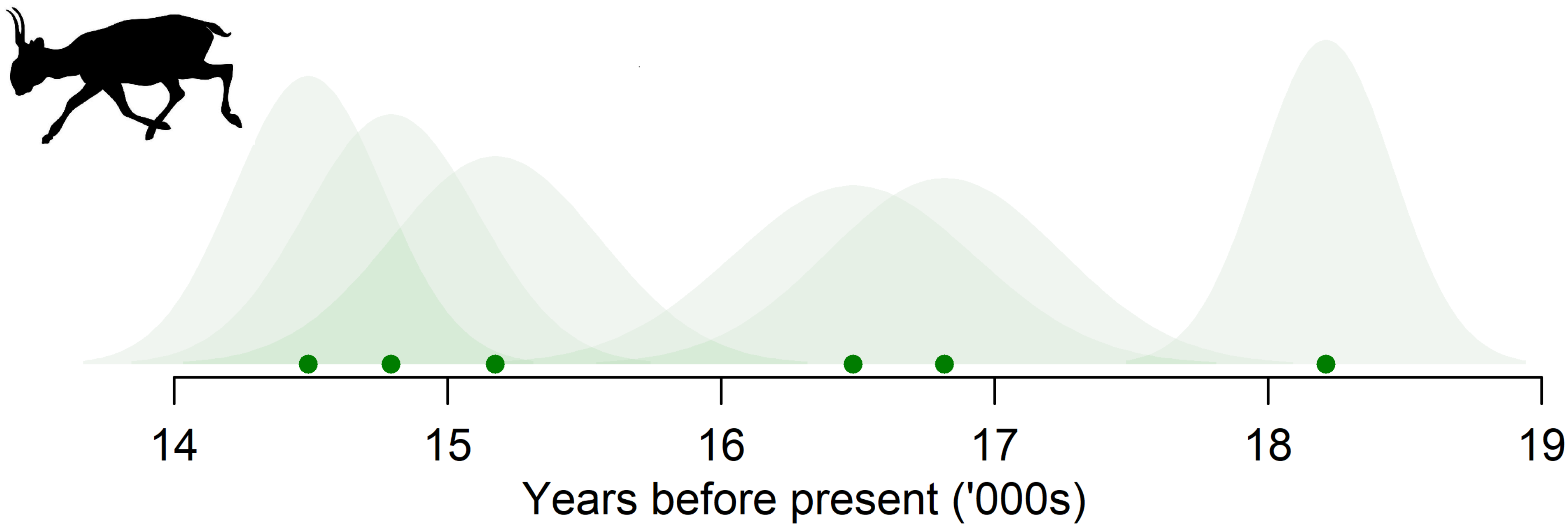
Problem: doesn't work for small σ (or small n)

… and simulations show asymptotic methods are no good for small σ and n.

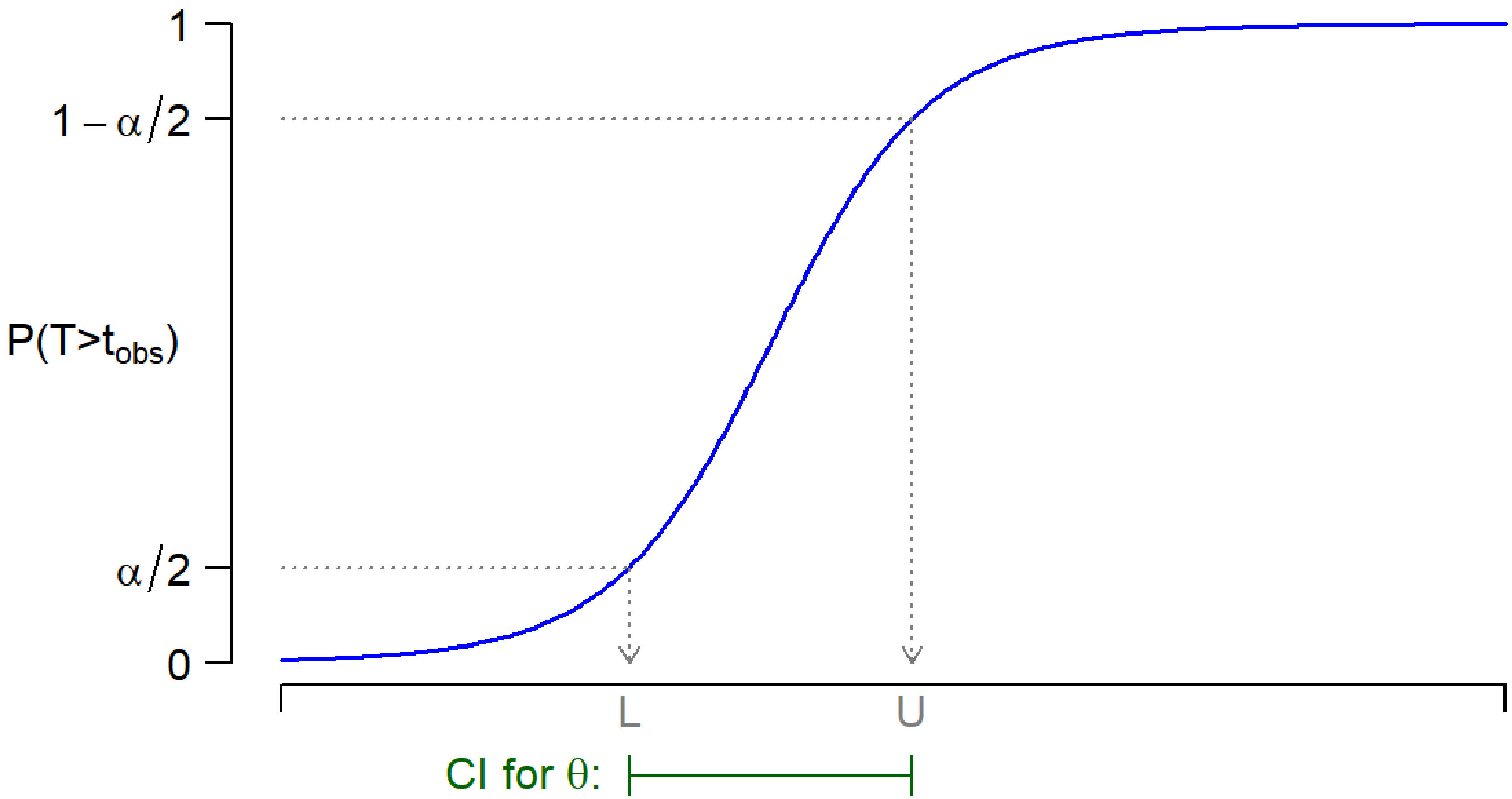


Problem: doesn't work for small σ (or small n)

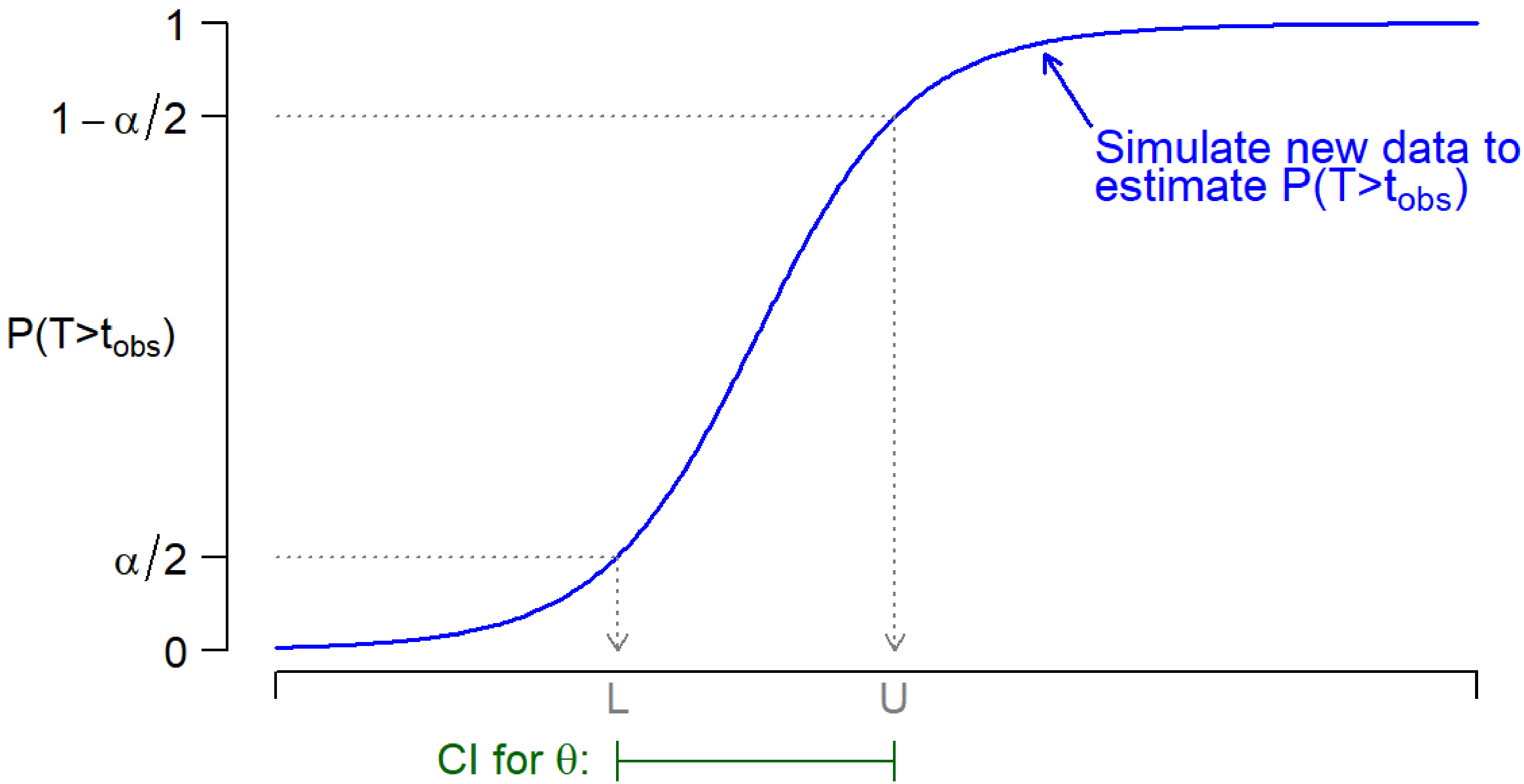
We usually have small σ and n!



Solution:



Solution: simulated (or “bootstrap”) inversion

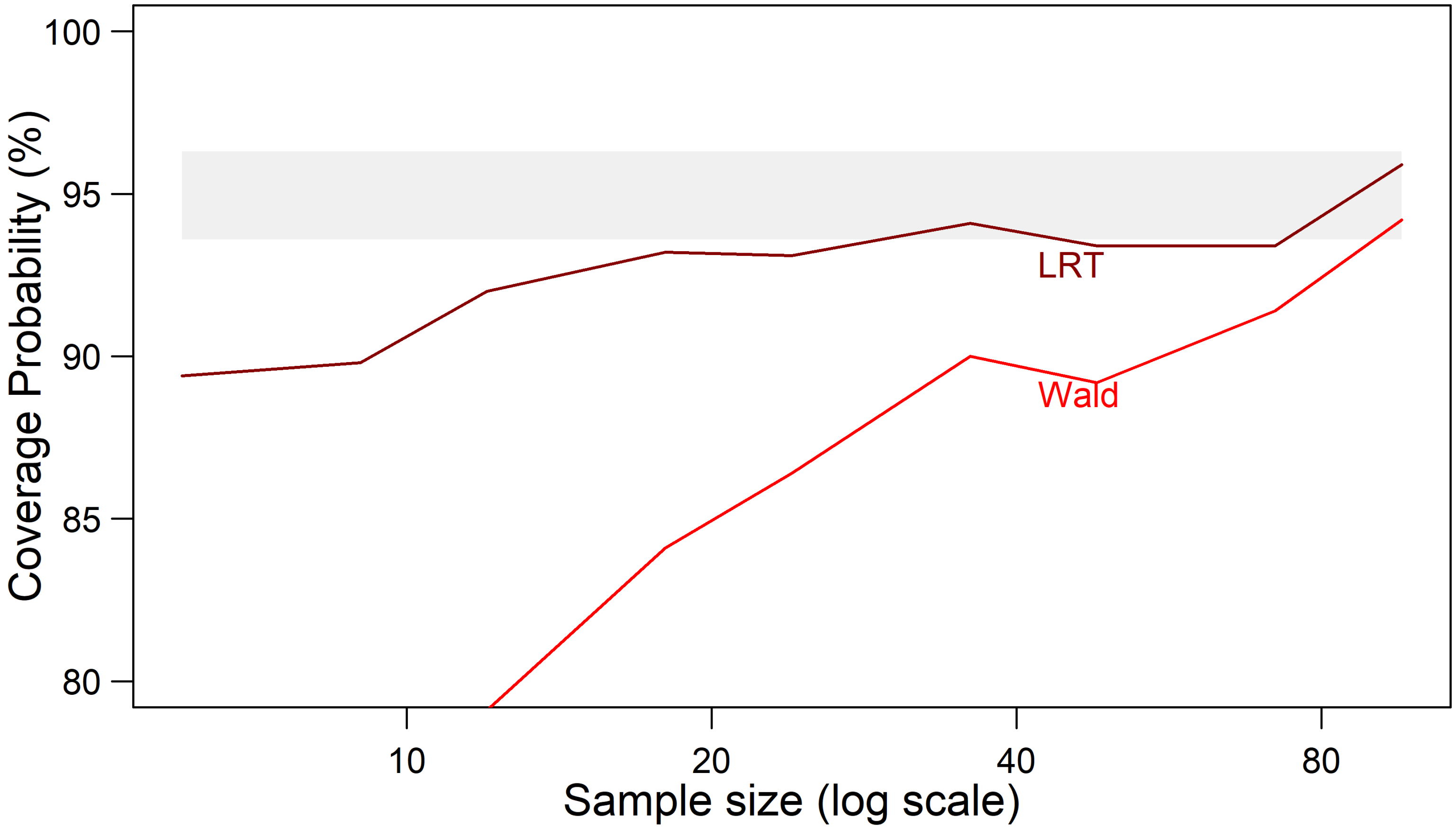


Simulated inversion

We tried two methods:

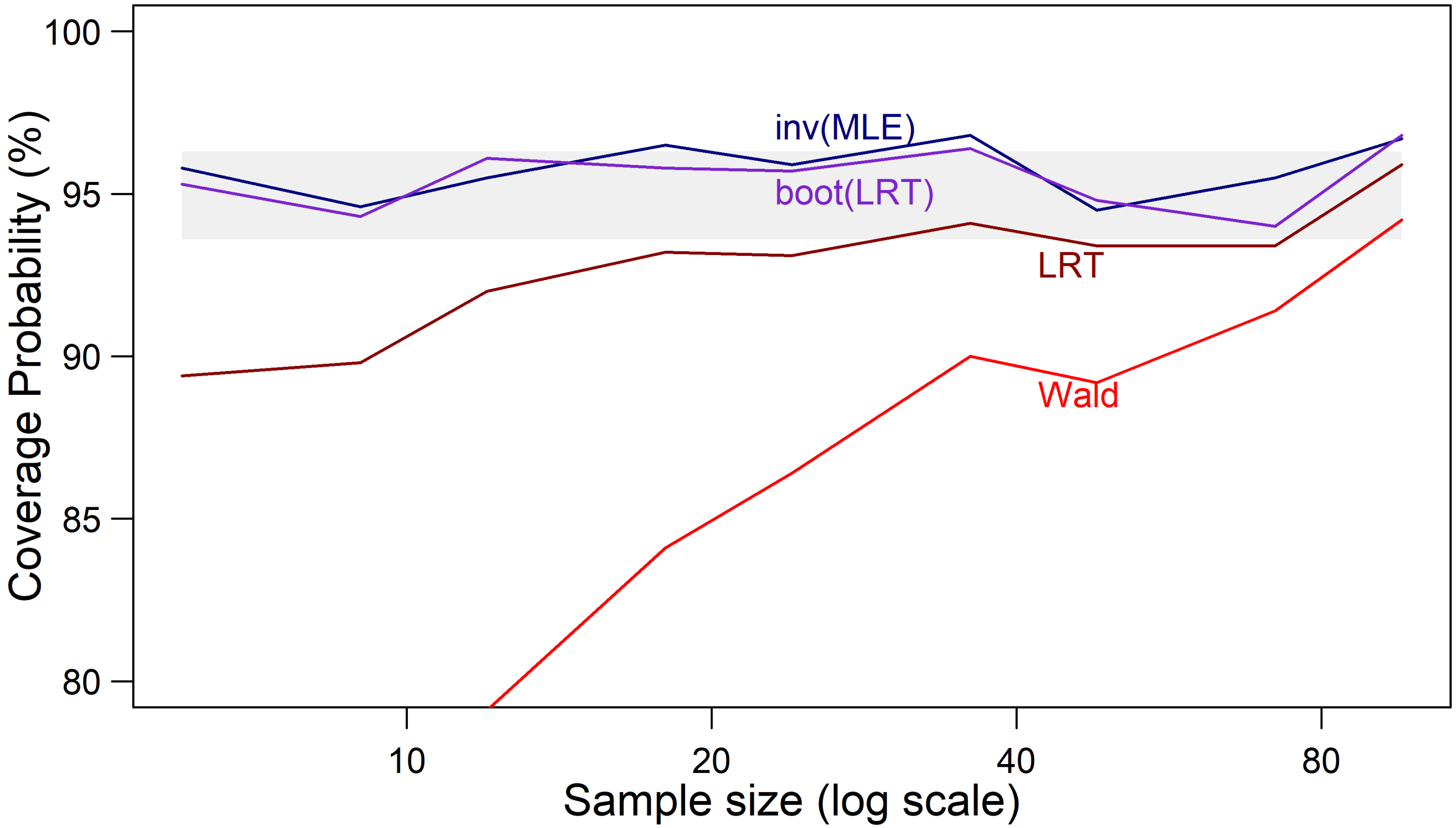
- `inv(MLE)` – use $T = \hat{\theta}$, get quantiles using quantile regression, simulate with θ set to current estimate. Similar to Fisher, Schweiger, and Rosset (2020).
- `boot(LRT)` – use $T = \mathrm{sign}(\hat{\theta} - \theta)\sqrt{-2\log\Lambda(\theta)}$, simulate at asymptotic estimates for CI limits and get sample quantiles (*parametric bootstrap*)

Results:

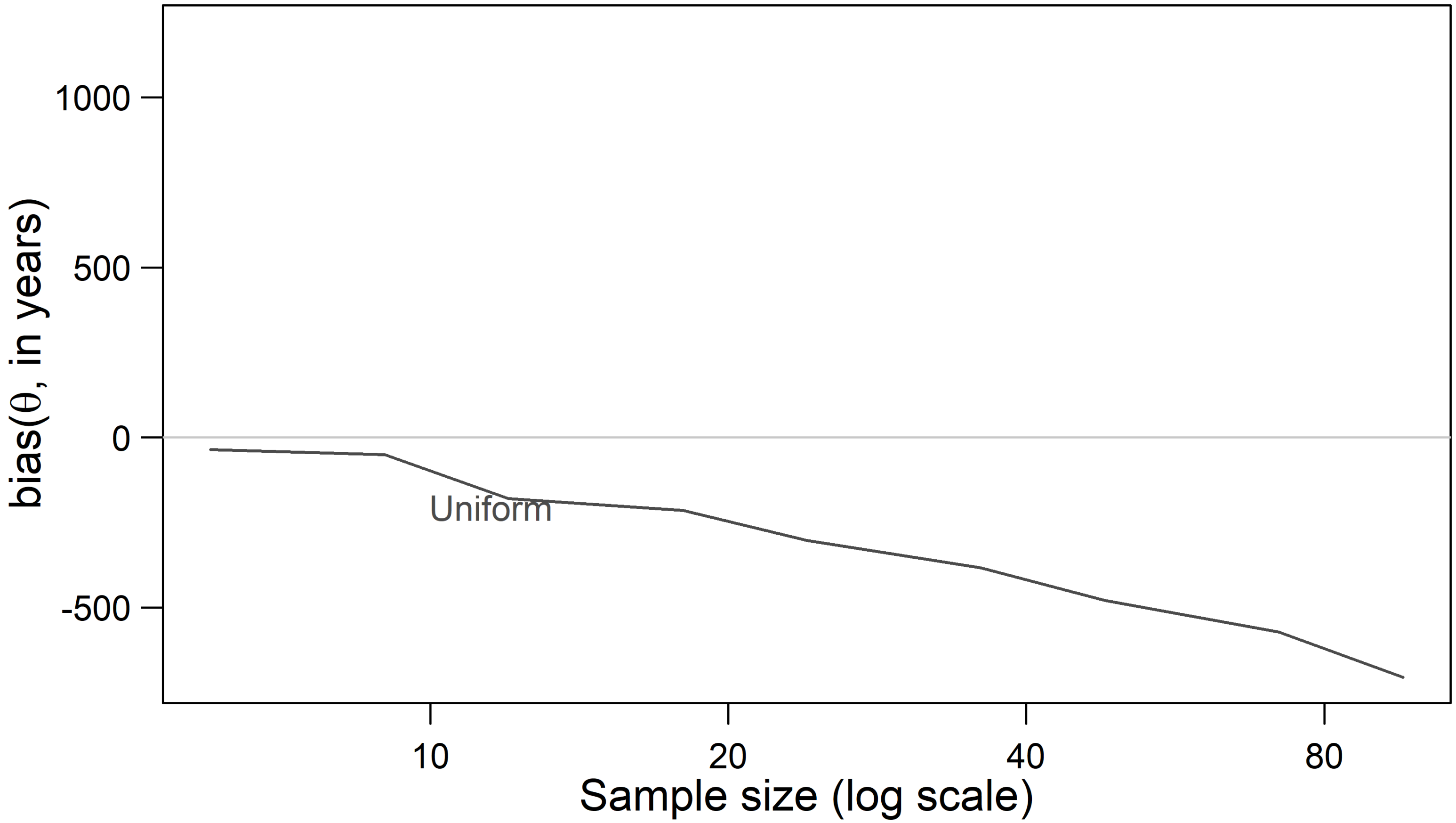


Coverage Probability (%)
100
95
90
85
80
LRT
Wald
10
20
40
80
Sample size (log scale)

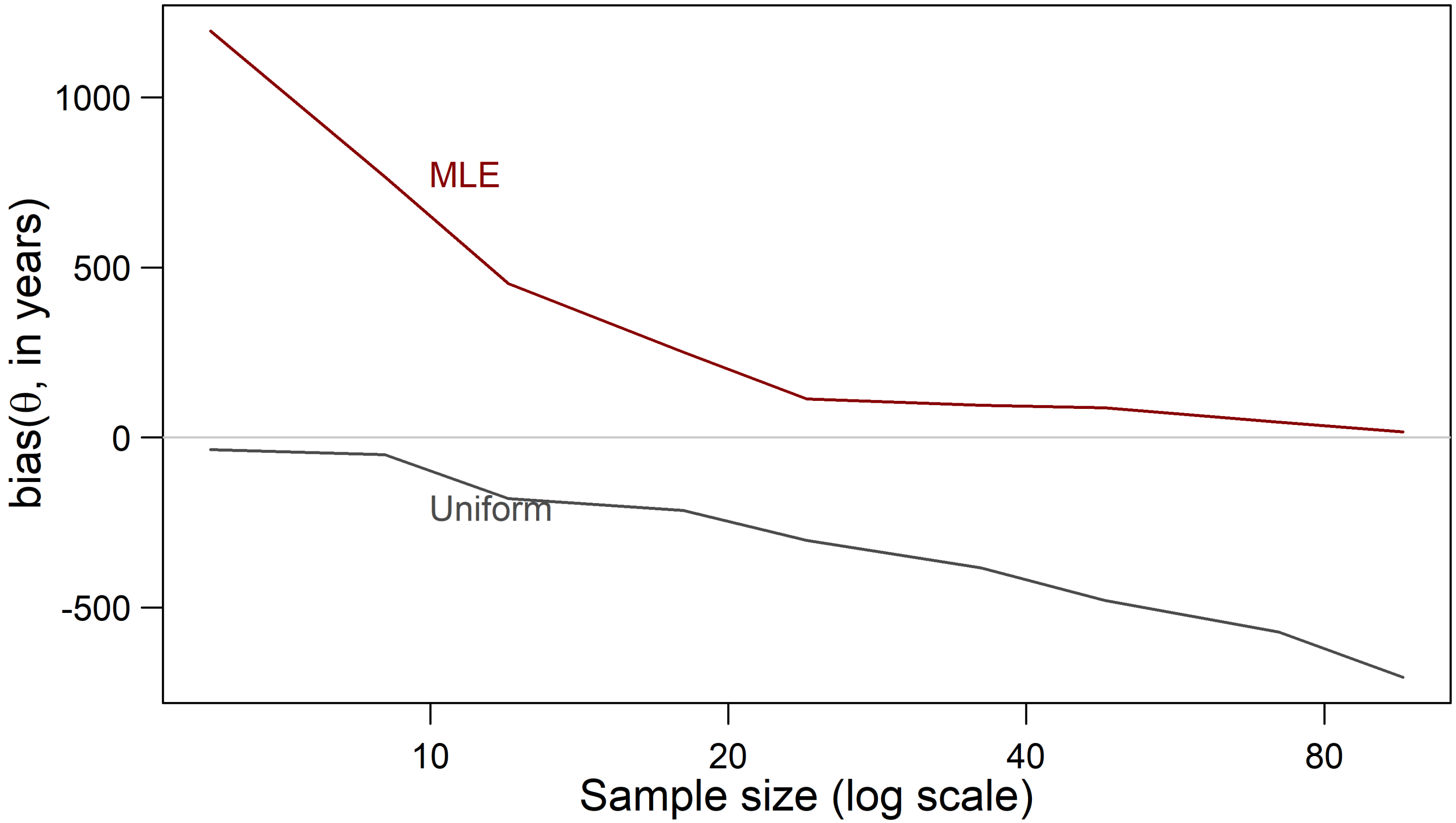
Results: good coverage for small σ or n



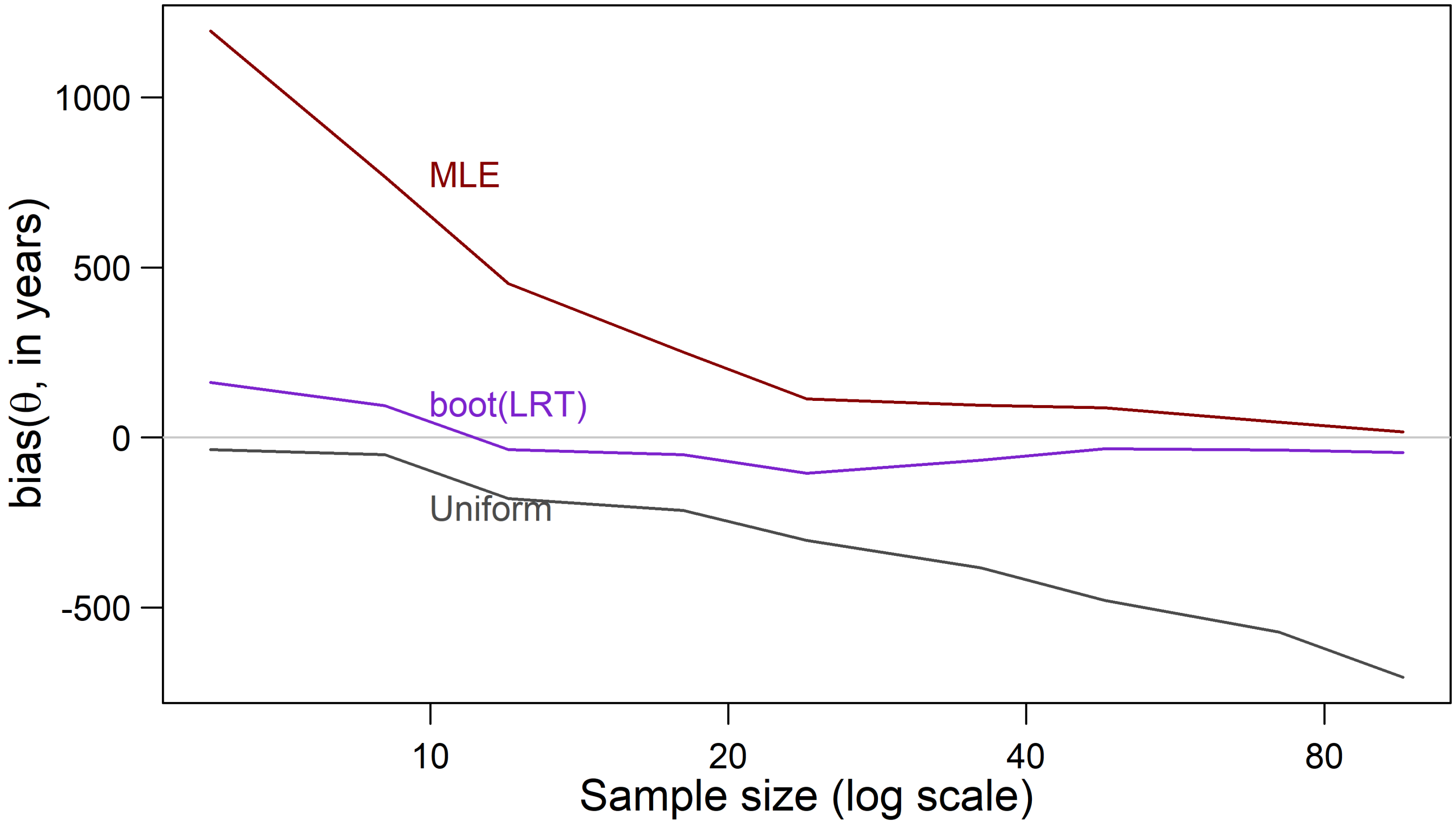
Results: bias



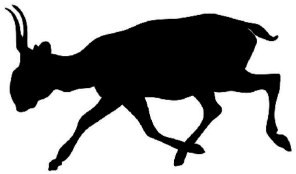
Results: bias correction needed…

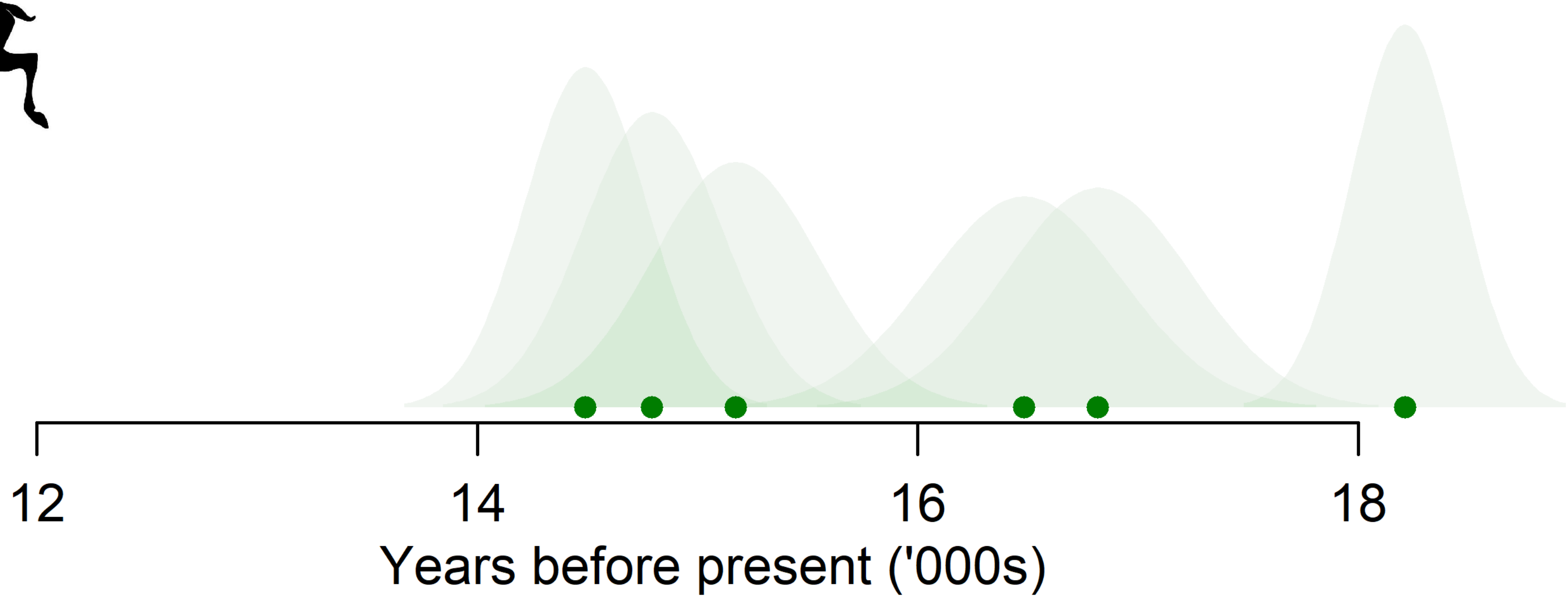


Results: bias correction needed…



Results: Saiga Antelope extinction estimates





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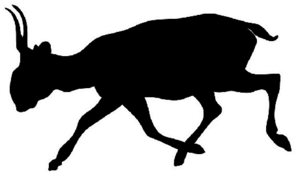

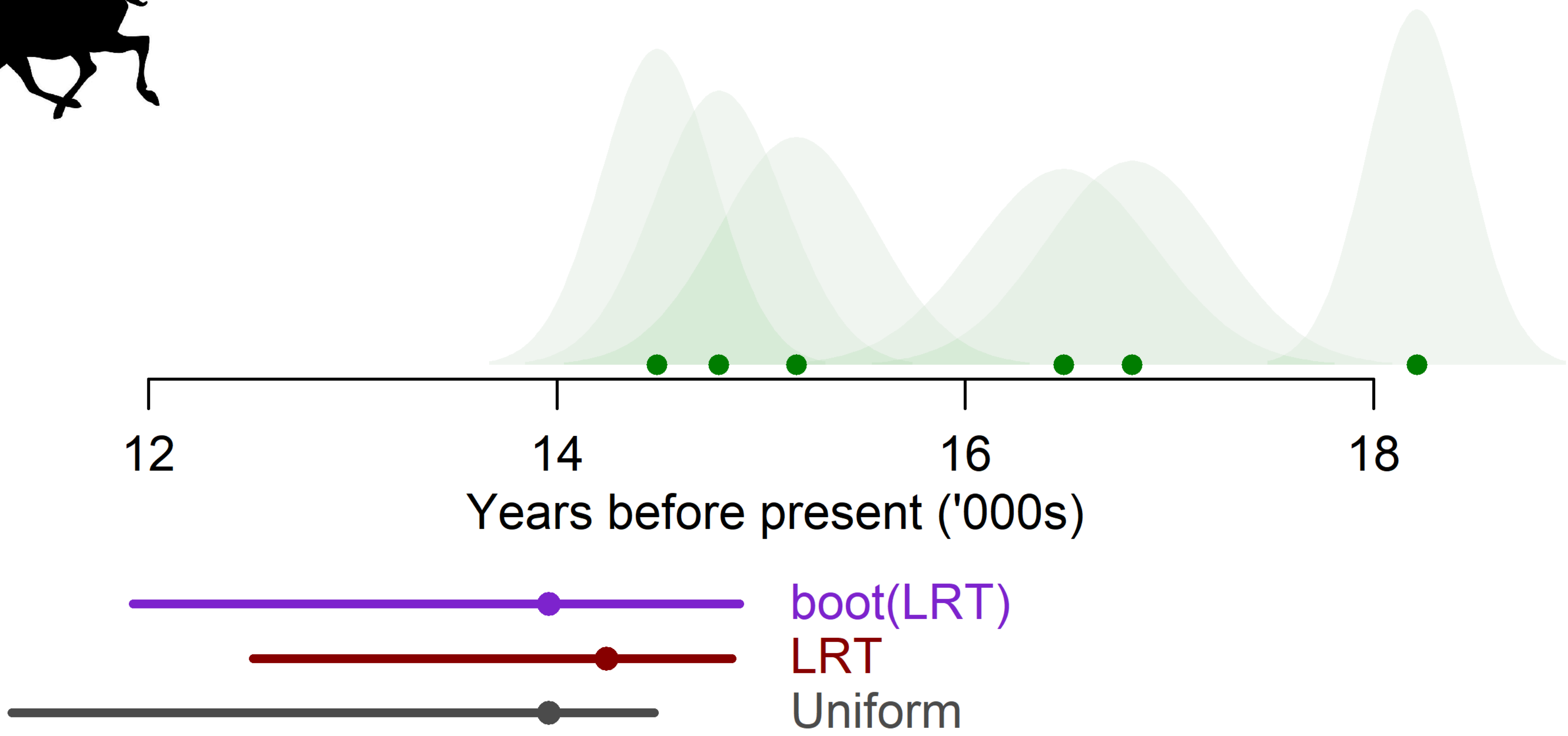

Conclusions

When estimating extinction time from the fossil record:

- You need to account for sampling and measurement error
- Common methods ignore measurement error $\rightarrow$ negative bias
- Classical asymptotics not a good idea $\rightarrow$ bootstrap from asymptotic estimates
- Future work to consider relaxing uniform assumption…

Acknowledgements

- Chris Turney and Alan Cooper for data
- UNSW Eco-Stats group for feedback
- UNSW School of Mathematics and Statistics

This research was undertaken on land traditionally owned by the Bedegal and Wallumedegal people, who I would like to acknowledge and pay my respect to, as well as to Aboriginal Elders past, present and emerging.

References

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